

FAST INERTIAL DYNAMICS AND FISTA ALGORITHMS IN CONVEX OPTIMIZATION. PERTURBATION ASPECTS.

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ABSTRACT. In a Hilbert space setting \mathcal{H} , we study the fast convergence properties as $t \rightarrow +\infty$ of the trajectories of the second-order differential equation

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = g(t),$$

where $\nabla\Phi$ is the gradient of a convex continuously differentiable function $\Phi : \mathcal{H} \rightarrow \mathbb{R}$, α is a positive parameter, and $g : [t_0, +\infty[\rightarrow \mathcal{H}$ is a "small" perturbation term. In this damped inertial system, the viscous damping coefficient $\frac{\alpha}{t}$ vanishes asymptotically, but not too rapidly. For $\alpha \geq 3$, and $\int_{t_0}^{+\infty} t\|g(t)\|dt < +\infty$, just assuming that $\operatorname{argmin} \Phi \neq \emptyset$, we show that any trajectory of the above system satisfies the fast convergence property

$$\Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq \frac{C}{t^2}.$$

For $\alpha > 3$, we show that any trajectory converges weakly to a minimizer of Φ , and we show the strong convergence property in various practical situations. This complements the results obtained by Su-Boyd- Candès, and Attouch-Peypouquet-Redont, in the unperturbed case $g = 0$. The parallel study of the time discretized version of this system provides new insight on the effect of errors, or perturbations on Nesterov's type algorithms. We obtain fast convergence of the values, and convergence of the trajectories for a perturbed version of the variant of FISTA recently considered by Chambolle-Dossal, and Su-Boyd-Candès.

1. INTRODUCTION

Throughout the paper, \mathcal{H} is a real Hilbert space which is endowed with the scalar product $\langle \cdot, \cdot \rangle$, with $\|x\|^2 = \langle x, x \rangle$ for any $x \in \mathcal{H}$. Let $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ be a convex differentiable function, whose gradient $\nabla\Phi$ is Lipschitz continuous on bounded sets. We suppose that $S = \operatorname{argmin} \Phi$ is nonempty. Let us give α a positive parameter. We are going to study the asymptotic behaviour (as $t \rightarrow +\infty$) of the trajectories of the second-order differential equation

$$(1) \quad (\text{AVD})_{\alpha, g} \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = g(t)$$

and consider similar questions for the corresponding algorithms. Let us give some $t_0 > 0$. The second-member $g : [t_0, +\infty[\rightarrow \mathcal{H}$ is a perturbation term (integrable source term), such that $g(t)$ is small for large t . Precisely, in our main result, Theorem 2.1, assuming that $\alpha \geq 3$, and $\int_{t_0}^{+\infty} t\|g(t)\|dt < +\infty$, we show that any trajectory of (1) satisfies the fast convergence property

$$(2) \quad \Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq \frac{C}{t^2}.$$

This extends the fast convergence of the values obtained by Su, Boyd and Candès in [41] in the unperturbed case $g = 0$. In Theorem 3.1, when $\alpha > 3$, we show that any trajectory of (1) converges weakly to a minimizer of Φ , which extends the convergence result obtained by Attouch, Peypouquet, and Redont in [15] in the case $g = 0$.

This inertial system involves a viscous damping which is attached to the term $\frac{\alpha}{t}\dot{x}(t)$. It is an isotropic linear damping with a viscous parameter $\frac{\alpha}{t}$ which vanishes asymptotically, but not too rapidly. The asymptotic behaviour of the inertial gradient-like system

$$(3) \quad (\text{AVD}) \quad \ddot{x}(t) + a(t)\dot{x}(t) + \nabla\Phi(x(t)) = 0,$$

with Asymptotic Vanishing Damping ((AVD) for short), has been studied by Cabot, Engler and Gaddat in [24]-[25]. As a main result, they proved that, under moderate decrease of $a(\cdot)$ to zero, i.e., $a(t) \rightarrow 0$ as $t \rightarrow +\infty$ with $\int_0^{+\infty} a(t)dt < +\infty$, then for any trajectory $x(\cdot)$ of (3)

$$(4) \quad \Phi(x(t)) \rightarrow \min_{\mathcal{H}} \Phi.$$

As a striking property, for the specific choice $a(t) = \frac{\alpha}{t}$, with $\alpha \geq 3$, for example when considering

$$(5) \quad \ddot{x}(t) + \frac{3}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0,$$

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it has been proved by Su, Boyd, and Candès in [41] that the fast convergence property of the values (2) is satisfied by the trajectories of (5). In the same article [41], the authors show that (5) can be seen as a continuous version of the fast convergent method of Nesterov, see [31]-[32]-[33]-[34]. For the continuous dynamic, a related study concerning the case $a(t) = \frac{1}{t^\theta}$, $0 < \theta < 1$ has been developed by Jendoubi and May in [29], with roughly speaking $\mathcal{O}(\frac{1}{t^{1+\theta}})$ convergence. The analysis developed in [29] does not contain the case $a(t) = \frac{\alpha}{t}$, where the introduction of an additional scaling, due to the coefficient α , requires a specific analysis. That's our main concern in this paper.

Our results provide new insight on the effect of perturbations or errors in the associated algorithms. They provide a guideline for the study of the preservation, under small perturbations, of the fast convergence property of the corresponding Nesterov type algorithms. Specifically we consider a perturbed version of the variant of FISTA recently considered by Chambolle and Dossal [26], and Su, Boyd and Candès [41]. We obtain fast convergence of the values in the case $\alpha \geq 3$, and convergence of the trajectories in the case $\alpha > 3$. Convergence of the trajectories in the case $\alpha = 3$, which corresponds to Nesterov algorithm, is still an open question.

2. FAST CONVERGENCE OF THE VALUES

Let $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ be a convex function, whose gradient $\nabla\Phi$ is Lipschitz continuous on bounded sets. Let $t_0 > 0$, $\alpha > 0$, and $g : [t_0, +\infty[\rightarrow \mathcal{H}$ such that $\int_{t_0}^{+\infty} \|g(t)\| dt < +\infty$. We consider the second-order differential equation

$$(6) \quad (\text{AVD})_{\alpha,g} \quad \ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + \nabla\Phi(x(t)) = g(t).$$

From Cauchy-Lipschitz theorem, for any Cauchy data $x(t_0) = x_0 \in \mathcal{H}$, $\dot{x}(t_0) = x_1 \in \mathcal{H}$ we immediately infer the existence and uniqueness of a local solution to (6). The global existence follows from the energy estimate proved in Proposition 2.1, in the next paragraph. Throughout this paper we will use the following Gronwall-Bellman lemma, see [20, Lemme A.5] for a proof.

Lemma 2.1. *Let $m \in L^1(t_0, T; \mathbb{R})$ such that $m \geq 0$ a.e. on $]t_0, T[$ and let c be a nonnegative constant. Suppose that w is a continuous function from $[t_0, T]$ into \mathbb{R} that satisfies, for all $t \in [t_0, T]$*

$$\frac{1}{2}w^2(t) \leq \frac{1}{2}c^2 + \int_{t_0}^t m(\tau)w(\tau)d\tau.$$

Then, for all $t \in [t_0, T]$

$$|w(t)| \leq c + \int_{t_0}^t m(\tau)d\tau.$$

2.1. Energy estimates. The following estimates are obtained by considering the global energy of the system, and showing that it is a strict Lyapunov function.

Proposition 2.1. *Suppose $\alpha > 0$, and $\int_{t_0}^{+\infty} \|g(t)\| dt < +\infty$. Then, for any orbit $x : [t_0, +\infty[\rightarrow \mathcal{H}$ of $(\text{AVD})_{\alpha,g}$*

$$(7) \quad \sup_t \|\dot{x}(t)\| < +\infty,$$

$$(8) \quad \int_{t_0}^{+\infty} \frac{1}{t} \|\dot{x}(t)\|^2 dt < +\infty.$$

Precisely, for any $t \geq t_0$

$$(9) \quad \|\dot{x}(t)\| \leq \|\dot{x}(t_0)\| + \sqrt{2} \left(\Phi(x_0) - \min_{\mathcal{H}} \Phi \right) + \int_{t_0}^{\infty} \|g(\tau)\| d\tau,$$

$$(10) \quad \int_{t_0}^{\infty} \frac{\alpha}{\tau} \|\dot{x}(\tau)\|^2 d\tau \leq \frac{1}{2} \|\dot{x}(t_0)\|^2 + \left(\Phi(x_0) - \min_{\mathcal{H}} \Phi \right) + \left(\|\dot{x}(t_0)\| + \sqrt{2} \left(\Phi(x_0) - \min_{\mathcal{H}} \Phi \right) + \|g\|_{L^1(t_0, \infty)} \right) \|g\|_{L^1(t_0, \infty)}.$$

Proof. Let us give some $T > t_0$. For $t_0 \leq t \leq T$, let us define the energy function

$$(11) \quad W_T(t) := \frac{1}{2} \|\dot{x}(t)\|^2 + \left(\Phi(x(t)) - \min_{\mathcal{H}} \Phi \right) + \int_t^T \langle \dot{x}(\tau), g(\tau) \rangle d\tau.$$

Because of \dot{x} continuous, and g integrable, the energy function W_T is well defined. After time derivation of W_T , and by using $(\text{AVD})_{\alpha,g}$, we obtain

$$\begin{aligned} \dot{W}_T(t) &:= \langle \dot{x}(t), \ddot{x}(t) + \nabla\Phi(x(t)) - g(t) \rangle \\ &= \langle \dot{x}(t), -\frac{\alpha}{t} \dot{x}(t) \rangle, \end{aligned}$$

that is

$$(12) \quad \dot{W}_T(t) + \frac{\alpha}{t} \|\dot{x}(t)\|^2 \leq 0.$$

Hence $W_T(\cdot)$ is a decreasing function. In particular, $W_T(t) \leq W_T(t_0)$, i.e.,

$$\frac{1}{2}\|\dot{x}(t)\|^2 + \left(\Phi(x(t)) - \min_{\mathcal{H}} \Phi\right) + \int_t^T \langle \dot{x}(\tau), g(\tau) \rangle d\tau \leq \frac{1}{2}\|\dot{x}(t_0)\|^2 + \left(\Phi(x_0) - \min_{\mathcal{H}} \Phi\right) + \int_{t_0}^T \langle \dot{x}(\tau), g(\tau) \rangle d\tau.$$

As a consequence,

$$\frac{1}{2}\|\dot{x}(t)\|^2 \leq \frac{1}{2}\|\dot{x}(t_0)\|^2 + \left(\Phi(x_0) - \min_{\mathcal{H}} \Phi\right) + \int_{t_0}^t \|\dot{x}(\tau)\| \|g(\tau)\| d\tau.$$

Applying Gronwall-Bellman lemma 2.1, we obtain

$$\begin{aligned} \|\dot{x}(t)\| &\leq \left(\|\dot{x}(t_0)\|^2 + 2\left(\Phi(x_0) - \min_{\mathcal{H}} \Phi\right)\right)^{\frac{1}{2}} + \int_{t_0}^t \|g(\tau)\| d\tau \\ &\leq \|\dot{x}(t_0)\| + \sqrt{2}\left(\Phi(x_0) - \min_{\mathcal{H}} \Phi\right) + \int_{t_0}^t \|g(\tau)\| d\tau. \end{aligned}$$

This being true for arbitrary $T > t_0$, and $t_0 \leq t \leq T$, we deduce that

$$(13) \quad \|\dot{x}(t)\| \leq \|\dot{x}(t_0)\| + \sqrt{2}\left(\Phi(x_0) - \min_{\mathcal{H}} \Phi\right) + \int_{t_0}^{\infty} \|g(\tau)\| d\tau,$$

which gives (7) and (9). As a consequence, the function W (corresponding to $T = +\infty$)

$$(14) \quad W(t) := \frac{1}{2}\|\dot{x}(t)\|^2 + \left(\Phi(x(t)) - \min_{\mathcal{H}} \Phi\right) + \int_t^{\infty} \langle \dot{x}(\tau), g(\tau) \rangle d\tau,$$

is well defined, and is minorized by

$$(15) \quad -\|\dot{x}\|_{L^\infty(t_0, +\infty)} \int_{t_0}^{\infty} \|g(\tau)\| d\tau.$$

By (12) we have

$$(16) \quad \dot{W}(t) + \frac{\alpha}{t}\|\dot{x}(t)\|^2 \leq 0.$$

Integrating (16) from t_0 to t , and using (13), (15), we obtain

$$\begin{aligned} \int_{t_0}^{\infty} \frac{\alpha}{\tau} \|\dot{x}(\tau)\|^2 d\tau &\leq \frac{1}{2}\|\dot{x}(t_0)\|^2 + \left(\Phi(x(t_0)) - \min_{\mathcal{H}} \Phi\right) + \|\dot{x}\|_{L^\infty(t_0, +\infty)} \int_{t_0}^{\infty} \|g(\tau)\| d\tau < +\infty \\ &\leq \frac{1}{2}\|\dot{x}(t_0)\|^2 + \left(\Phi(x(t_0)) - \min_{\mathcal{H}} \Phi\right) + \left(\|\dot{x}(t_0)\| + \sqrt{2}\left(\Phi(x_0) - \min_{\mathcal{H}} \Phi\right) + \|g\|_{L^1(t_0, +\infty)}\right) \|g\|_{L^1(t_0, +\infty)}, \end{aligned}$$

which gives (8) and (10). \square

2.2. The main result. Let us state our main result.

Theorem 2.1. *Suppose that $\alpha \geq 3$, and $\int_{t_0}^{+\infty} \tau \|g(\tau)\| d\tau < +\infty$. Then, for any orbit $x : [t_0, +\infty[\rightarrow \mathcal{H}$ of $(AVD)_{\alpha, g}$, we have the following fast convergence of the values:*

$$\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}\left(\frac{1}{t^2}\right).$$

Precisely

$$(17) \quad \frac{2}{\alpha-1}t^2(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) \leq C + 2\left(\left(\frac{C}{\alpha-1}\right)^{\frac{1}{2}} + \frac{1}{\alpha-1} \int_{t_0}^{\infty} \tau \|g(\tau)\| d\tau\right) \int_{t_0}^{\infty} \tau \|g(\tau)\| d\tau,$$

with

$$C = \frac{2}{\alpha-1}t^2(\Phi(x_0) - \inf_{\mathcal{H}} \Phi) + (\alpha-1)\|x_0 - x^*\|^2 + \frac{t_0}{\alpha-1}\|\dot{x}(t_0)\|^2.$$

Moreover

$$(18) \quad \sup_{t \geq t_0} \left\|x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t)\right\| \leq \left(\frac{C}{\alpha-1}\right)^{\frac{1}{2}} + \frac{1}{\alpha-1} \int_{t_0}^{\infty} \tau \|g(\tau)\| d\tau < +\infty.$$

Proof. The proof is an adaptation to our setting (with an integrable source term g) of the argument developed by Su-Boyd-Candès in [41]. Let us give some $T > t_0$, and $x^* \in S = \operatorname{argmin} \Phi$. For $t_0 \leq t \leq T$, let us define the energy function

$$(19) \quad \mathcal{E}_{\alpha, g, T}(t) := \frac{2}{\alpha-1}t^2(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) + (\alpha-1)\|x(t) - x^*\|^2 + \frac{t}{\alpha-1}\|\dot{x}(t)\|^2 + 2 \int_t^T \tau \langle x(\tau) - x^* + \frac{\tau}{\alpha-1}\dot{x}(\tau), g(\tau) \rangle d\tau.$$

Let us show that

$$\dot{\mathcal{E}}_{\alpha, g, T}(t) + 2\frac{\alpha-3}{\alpha-1}t(\Phi(x(t)) - \min_{\mathcal{H}} \Phi) \leq 0.$$

Derivation of $\mathcal{E}_{\alpha,g,T}(\cdot)$ gives

$$\begin{aligned}\dot{\mathcal{E}}_{\alpha,g,T}(t) &:= \frac{4}{\alpha-1}t(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) + \frac{2}{\alpha-1}t^2\langle \nabla\Phi(x(t)), \dot{x}(t) \rangle \\ &\quad + 2(\alpha-1)\langle x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t), \dot{x}(t) + \frac{1}{\alpha-1}\dot{x}(t) + \frac{t}{\alpha-1}\ddot{x}(t) \rangle - 2t\langle x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t), g(t) \rangle \\ &= \frac{4}{\alpha-1}t(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) + \frac{2}{\alpha-1}t^2\langle \nabla\Phi(x(t)), \dot{x}(t) \rangle \\ &\quad + 2(\alpha-1)\langle x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t), \frac{t}{\alpha-1}\left(\frac{\alpha}{t}\dot{x}(t) + \ddot{x}(t) - g(t)\right) \rangle.\end{aligned}$$

Then use $(\text{AVD})_{\alpha,g}$ in this last expression to obtain

$$(20) \quad \dot{\mathcal{E}}_{\alpha,g,T}(t) = \frac{4}{\alpha-1}t(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) + \frac{2}{\alpha-1}t^2\langle \nabla\Phi(x(t)), \dot{x}(t) \rangle$$

$$(21) \quad -2t\langle x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t), \nabla\Phi(x(t)) \rangle$$

$$(22) \quad = \frac{4}{\alpha-1}t(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) - 2t\langle x(t) - x^*, \nabla\Phi(x(t)) \rangle.$$

By convexity of Φ

$$\Phi(x^*) \geq \Phi(x(t)) + \langle x^* - x(t), \nabla\Phi(x(t)) \rangle.$$

Replacing in (22) we obtain

$$\dot{\mathcal{E}}_{\alpha,g,T}(t) + \left(2 - \frac{4}{\alpha-1}\right)t(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) \leq 0.$$

Equivalently

$$(23) \quad \dot{\mathcal{E}}_{\alpha,g,T}(t) + 2\frac{\alpha-3}{\alpha-1}t(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) \leq 0.$$

As a consequence, for $\alpha \geq 3$, the function $\mathcal{E}_{\alpha,g}$ is nonincreasing. In particular, $\mathcal{E}_{\alpha,g}(t) \leq \mathcal{E}_{\alpha,g}(t_0)$, which gives

$$\begin{aligned}&\frac{2}{\alpha-1}t^2(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) + (\alpha-1)\|x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t)\|^2 + 2\int_t^T \tau\langle x(\tau) - x^* + \frac{\tau}{\alpha-1}\dot{x}(\tau), g(\tau) \rangle d\tau \\ &\leq \frac{2}{\alpha-1}t_0^2(\Phi(x_0) - \inf_{\mathcal{H}}\Phi) + (\alpha-1)\|x_0 - x^* + \frac{t_0}{\alpha-1}\dot{x}(t_0)\|^2 + 2\int_{t_0}^T \tau\langle x(\tau) - x^* + \frac{\tau}{\alpha-1}\dot{x}(\tau), g(\tau) \rangle d\tau.\end{aligned}$$

Equivalently

$$(24) \quad \frac{2}{\alpha-1}t^2(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) + (\alpha-1)\|x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t)\|^2 \leq C + 2\int_{t_0}^t \tau\langle x(\tau) - x^* + \frac{\tau}{\alpha-1}\dot{x}(\tau), g(\tau) \rangle d\tau,$$

with

$$C = \frac{2}{\alpha-1}t_0^2(\Phi(x_0) - \inf_{\mathcal{H}}\Phi) + (\alpha-1)\|x_0 - x^* + \frac{t_0}{\alpha-1}\dot{x}(t_0)\|^2.$$

From (24) we infer

$$(25) \quad \frac{1}{2}\|x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t)\|^2 \leq \frac{C}{2(\alpha-1)} + \frac{1}{\alpha-1}\int_{t_0}^t \|x(\tau) - x^* + \frac{\tau}{\alpha-1}\dot{x}(\tau)\| \tau g(\tau) d\tau.$$

Applying once more Gronwall-Bellman lemma 2.1, we obtain

$$(26) \quad \|x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t)\| \leq \left(\frac{C}{\alpha-1}\right)^{\frac{1}{2}} + \frac{1}{\alpha-1}\int_{t_0}^t \tau\|g(\tau)\| d\tau.$$

Since $\int_{t_0}^{+\infty} t\|g(t)\| dt < +\infty$, it follows that

$$(27) \quad \sup_t \|x(t) - x^* + \frac{t}{\alpha-1}\dot{x}(t)\| \leq \left(\frac{C}{\alpha-1}\right)^{\frac{1}{2}} + \frac{1}{\alpha-1}\int_{t_0}^{\infty} \tau\|g(\tau)\| d\tau < +\infty.$$

Returning to (24), we conclude that

$$(28) \quad \frac{2}{\alpha-1}t^2(\Phi(x(t)) - \inf_{\mathcal{H}}\Phi) \leq C + 2\left(\left(\frac{C}{\alpha-1}\right)^{\frac{1}{2}} + \frac{1}{\alpha-1}\int_{t_0}^{\infty} \tau\|g(\tau)\| d\tau\right)\int_{t_0}^{\infty} \tau\|g(\tau)\| d\tau.$$

□

Remark 2.1. As a consequence the energy function

$$(29) \quad \mathcal{E}_{\alpha,g}(t) := \frac{2}{\alpha-1} t^2 (\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) + (\alpha-1) \|x(t) - x^* + \frac{t}{\alpha-1} \dot{x}(t)\|^2 + 2 \int_t^{+\infty} \tau \langle x(\tau) - x^* + \frac{\tau}{\alpha-1} \dot{x}(\tau), g(\tau) \rangle d\tau.$$

is well defined, and is a Lyapunov function for the dynamical system $(AVD)_{\alpha,g}$.

3. CONVERGENCE OF TRAJECTORIES

In the case $\alpha > 3$, provided that the second member $g(t)$ is sufficiently small for large t , we are going to show the convergence of the trajectories of the system

$$(AVD)_{\alpha,g} \quad \ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + \nabla \Phi(x(t)) = g(t).$$

3.1. Main statement, and preliminary results. The following convergence result is an extension to the perturbed case (with a source term g) of the convergence result obtained by Attouch-Peypouquet-Redont in [15].

Theorem 3.1. *Let $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ a convex continuously differentiable function such that $S = \operatorname{argmin} \Phi$ is nonempty. Suppose that $\alpha > 3$ and $\int_{t_0}^{+\infty} t \|g(t)\| dt < +\infty$. Let $t_0 > 0$, and $x : [t_0, +\infty[\rightarrow \mathcal{H}$ be a classical solution of $(AVD)_{\alpha,g}$. Then, the following convergence properties hold:*

a) (weak convergence) *There exists some $x^* \in \operatorname{argmin} \Phi$ such that*

$$(30) \quad x(t) \rightharpoonup x^* \text{ weakly as } t \rightarrow +\infty.$$

b) (fast convergence) *There exists a positive constant C such that*

$$(31) \quad \Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq \frac{C}{t^2}$$

$$(32) \quad \int_{t_0}^{\infty} t \left(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi \right) dt < +\infty.$$

c) (energy estimate)

$$(33) \quad \int_{t_0}^{\infty} t \|\dot{x}(t)\|^2 dt < +\infty$$

$$(34) \quad \|\dot{x}(t)\| \leq \frac{C}{t}$$

and hence

$$(35) \quad \lim_{t \rightarrow \infty} \|\dot{x}(t)\| = 0.$$

In order to analyze the convergence properties of the trajectories of system (1), we will use the Opial's lemma [35] that we recall in its continuous form; see also [22], who initiated the use of this argument to analyze the asymptotic convergence of nonlinear contraction semigroups in Hilbert spaces.

Lemma 3.1. *Let S be a non empty subset of \mathcal{H} and $x : [0, +\infty[\rightarrow \mathcal{H}$ a map. Assume that*

- (i) *for every $z \in S$, $\lim_{t \rightarrow +\infty} \|x(t) - z\|$ exists;*
- (ii) *every weak sequential cluster point of the map x belongs to S .*

Then

$$w - \lim_{t \rightarrow +\infty} x(t) = x_{\infty} \quad \text{exists, for some element } x_{\infty} \in S.$$

We also need the following result concerning the integration of a first-order nonautonomous differential inequation, see [15].

Lemma 3.2. *Suppose that $\delta > 0$, and let $w : [\delta, +\infty[\rightarrow \mathbb{R}$ be a continuously differentiable function that satisfies the following differential inequality*

$$(36) \quad \dot{w}(t) + \frac{\alpha}{t} w(t) \leq k(t),$$

for some $\alpha > 1$, and some nonnegative function $k : [\delta, +\infty[\rightarrow \mathbb{R}$ such that $t \mapsto tk(t) \in L^1(\delta, +\infty)$. Then

$$(37) \quad w^+ \in L^1(\delta, +\infty).$$

3.2. Proof of the convergence results.

Proof. Step 1. Let us return to the decrease property (23) which is satisfied by the Lyapunov function $\mathcal{E}_{\alpha,g}$:

$$\dot{\mathcal{E}}_{\alpha,g}(t) + 2\frac{\alpha-3}{\alpha-1}t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) \leq 0.$$

By integration of this inequality, we obtain

$$\mathcal{E}_{\alpha,g}(t) + 2\frac{\alpha-3}{\alpha-1} \int_{t_0}^t \tau(\Phi(x(\tau)) - \inf_{\mathcal{H}} \Phi) d\tau \leq \mathcal{E}_{\alpha,g}(t_0).$$

By definition of $\mathcal{E}_{\alpha,g}$, and neglecting its nonnegative terms, we infer

$$2 \int_t^{+\infty} \tau \langle x(\tau) - x^* + \frac{\tau}{\alpha-1} \dot{x}(\tau), g(\tau) \rangle d\tau + 2\frac{\alpha-3}{\alpha-1} \int_{t_0}^t \tau(\Phi(x(\tau)) - \inf_{\mathcal{H}} \Phi) d\tau \leq \mathcal{E}_{\alpha,g}(t_0).$$

Hence

$$2\frac{\alpha-3}{\alpha-1} \int_{t_0}^t \tau(\Phi(x(\tau)) - \inf_{\mathcal{H}} \Phi) d\tau \leq \mathcal{E}_{\alpha,g}(t_0) + 2 \int_{t_0}^{+\infty} \|x(\tau) - x^* + \frac{\tau}{\alpha-1} \dot{x}(\tau)\| \|\tau g(\tau)\| d\tau.$$

By (27), we have

$$\sup_t \|x(t) - x^* + \frac{t}{\alpha-1} \dot{x}(t)\| < +\infty.$$

As a consequence

$$2\frac{\alpha-3}{\alpha-1} \int_{t_0}^t \tau(\Phi(x(\tau)) - \inf_{\mathcal{H}} \Phi) d\tau \leq \mathcal{E}_{\alpha,g}(t_0) + 2 \sup_t \|x(t) - x^* + \frac{t}{\alpha-1} \dot{x}(t)\| \int_{t_0}^{+\infty} \|\tau g(\tau)\| d\tau.$$

Since $\alpha > 3$, we deduce that

$$(38) \quad \int_{t_0}^{+\infty} \tau(\Phi(x(\tau)) - \inf_{\mathcal{H}} \Phi) d\tau < +\infty.$$

Step 2. Let us show that

$$\int_{t_0}^{\infty} t \|\dot{x}(t)\|^2 dt < +\infty.$$

To that end, we use the energy estimate which is obtained by taking the scalar product of (1) by $t^2 \dot{x}(t)$:

$$(39) \quad t^2 \langle \ddot{x}(t), \dot{x}(t) \rangle + \alpha t \|\dot{x}(t)\|^2 + t^2 \langle \nabla \Phi(x(t)), \dot{x}(t) \rangle = t^2 \langle g(t), \dot{x}(t) \rangle.$$

By the classical derivation chain rule, and Cauchy-Schwarz inequality, we obtain

$$(40) \quad \frac{1}{2} t^2 \frac{d}{dt} \|\dot{x}(t)\|^2 + \alpha t \|\dot{x}(t)\|^2 + t^2 \frac{d}{dt} \Phi(x(t)) \leq \|tg(t)\| \|t\dot{x}(t)\|.$$

After integration by parts

$$\begin{aligned} & \frac{t^2}{2} \|\dot{x}(t)\|^2 - \frac{t_0^2}{2} \|\dot{x}(t_0)\|^2 - \int_{t_0}^t s \|\dot{x}(s)\|^2 ds + \alpha \int_{t_0}^t s \|\dot{x}(s)\|^2 ds \\ & + t^2 (\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) - t_0^2 (\Phi(x(t_0)) - \inf_{\mathcal{H}} \Phi) - 2 \int_{t_0}^t s (\Phi(x(s)) - \inf_{\mathcal{H}} \Phi) ds \leq \int_{t_0}^t \|sg(s)\| \|s\dot{x}(s)\| ds. \end{aligned}$$

As a consequence, for some constant $C \geq 0$, depending only on the Cauchy data,

$$(41) \quad \frac{t^2}{2} \|\dot{x}(t)\|^2 + (\alpha-1) \int_{t_0}^t s \|\dot{x}(s)\|^2 ds \leq C + 2 \int_{t_0}^t s (\Phi(x(s)) - \inf_{\mathcal{H}} \Phi) ds + \int_{t_0}^t \|sg(s)\| \|s\dot{x}(s)\| ds.$$

By (38) we have $\int_{t_0}^{\infty} s (\Phi(x(s)) - \inf_{\mathcal{H}} \Phi) ds < +\infty$. Moreover $\alpha > 1$. As a consequence, from (41) we deduce that, for some other constant C

$$(42) \quad \frac{1}{2} \|t\dot{x}(t)\|^2 \leq C + \int_{t_0}^t \|sg(s)\| \|s\dot{x}(s)\| ds.$$

Applying Gronwall-Bellman lemma 2.1, we obtain

$$\|t\dot{x}(t)\| \leq \sqrt{2C} + \int_{t_0}^t \|sg(s)\| ds.$$

Since $\int_{t_0}^{+\infty} t \|g(t)\| dt < +\infty$, we infer

$$(43) \quad \sup_t \|t\dot{x}(t)\| < +\infty.$$

Returning to (41), we deduce that

$$(44) \quad (\alpha - 1) \int_{t_0}^t s \|\dot{x}(s)\|^2 ds \leq C + 2 \int_{t_0}^{\infty} s(\Phi(x(s)) - \inf_{\mathcal{H}} \Phi) ds + \sup_t \|\dot{x}(t)\| \int_{t_0}^{\infty} \|sg(s)\| ds,$$

which gives

$$\int_{t_0}^{\infty} t \|\dot{x}(t)\|^2 dt < +\infty.$$

Moreover, combining (27),

$$\sup_t \|x(t) - x^* + \frac{t}{\alpha - 1} \dot{x}(t)\| < +\infty,$$

with (43), we deduce that

$$(45) \quad \sup_t \|x(t)\| < +\infty,$$

i.e., all the orbits are bounded.

Step 3. Our proof of the weak convergence property of the orbits of $(\text{AVD})_{\alpha, g}$ relies on Opial's lemma. Given $x^* \in \text{argmin } \Phi$, let us define $h : [0, +\infty[\rightarrow \mathbb{R}^+$ by

$$(46) \quad h(t) = \frac{1}{2} \|x(t) - x^*\|^2.$$

By the classical derivation chain rule

$$(47) \quad \dot{h}(t) = \langle x(t) - x^*, \dot{x}(t) \rangle,$$

$$(48) \quad \ddot{h}(t) = \langle x(t) - x^*, \ddot{x}(t) \rangle + \|\dot{x}(t)\|^2.$$

Combining these two equations, and using (1) we obtain

$$(49) \quad \ddot{h}(t) + \frac{\alpha}{t} \dot{h}(t) = \|\dot{x}(t)\|^2 + \langle x(t) - x^*, \ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) \rangle,$$

$$(50) \quad = \|\dot{x}(t)\|^2 + \langle x(t) - x^*, -\nabla \Phi(x(t)) + g(t) \rangle.$$

By monotonicity of $\nabla \Phi$ and $\nabla \Phi(x^*) = 0$

$$(51) \quad \langle x(t) - x^*, -\nabla \Phi(x(t)) \rangle \leq 0.$$

By (49) and (51) we infer

$$(52) \quad \ddot{h}(t) + \frac{\alpha}{t} \dot{h}(t) \leq \|\dot{x}(t)\|^2 + \|x(t) - x^*\| \|g(t)\|.$$

Equivalently

$$(53) \quad \ddot{h}(t) + \frac{\alpha}{t} \dot{h}(t) \leq k(t),$$

with

$$k(t) := \|\dot{x}(t)\|^2 + \|x(t) - x^*\| \|g(t)\|.$$

By (45) the orbit is bounded. Hence, for some constant $C \geq 0$

$$k(t) \leq \|\dot{x}(t)\|^2 + C \|g(t)\|.$$

By assumption $\int_{t_0}^{+\infty} t \|g(t)\| dt < +\infty$, and by (33) $\int_{t_0}^{\infty} t \|\dot{x}(t)\|^2 dt < +\infty$. Hence $t \mapsto tk(t) \in L^1(t_0, +\infty)$. Applying Lemma 3.2, with $w(t) = \dot{h}(t)$, we deduce that $w^+ \in L^1(t_0, +\infty)$. Equivalently $\dot{h}^+(t) \in L^1(t_0, +\infty)$, which implies that the limit of $h(t)$ exists, as $t \rightarrow +\infty$. This proves item *i*) of the Opial's lemma. We complete the proof by observing that item *ii*) is satisfied too. Indeed, since $\Phi(x(t))$ converges to $\inf \Phi$, we have that every weak sequential cluster point of $x(\cdot)$ is a minimizer of Φ . \square

3.3. Strong convergence results. Since the work of J.B. Baillon, we know that without additional assumptions, the trajectories of the gradient systems may not converge strongly. Let's examine some practical interest situations where strong convergence of the trajectories of $(\text{AVD})_{\alpha, g}$ is satisfied.

Strong convergence under $\text{int}(\text{argmin } \Phi) \neq \emptyset$.

We will need the following result, see ([15], Lemma 5.4).

Lemma 3.3. *Suppose that $\delta > 0$, and let $f : [\delta; +\infty[\rightarrow \mathcal{H}$ be a continuous function that satisfies $f \in L^1(\delta; +\infty; \mathcal{H})$. Suppose that $\alpha > 1$ and $x : [\delta; +\infty[\rightarrow \mathcal{H}$ is a classical solution of*

$$t\ddot{x}(t) + \alpha\dot{x}(t) = f(t).$$

Then, $x(t)$ converges strongly in \mathcal{H} as $t \rightarrow \infty$.

Theorem 3.2. *Suppose that $\alpha > 3$, $\int_{t_0}^{+\infty} tg(t)dt < +\infty$, and Φ satisfies $\text{int}(\text{argmin } \Phi) \neq \emptyset$. Let $x(\cdot)$ be a classical global solution of equation (1). Then, there exists some $x^* \in \text{argmin } \Phi$ such that $x(t) \rightarrow x^*$ strongly as $t \rightarrow +\infty$.*

Proof. We follow the same approach as that proposed in [15, Theorem 3.1]. We first observe that the assumption $\text{int}(\text{argmin } \Phi) \neq \emptyset$ implies the existence of some $\bar{z} \in \mathcal{H}$ and $\rho > 0$ such that, for all $x \in \mathcal{H}$, $\langle \nabla \Phi(x), x - \bar{z} \rangle \geq \rho \|\nabla \Phi(x)\|$. In particular, for all $t \geq t_0$

$$\langle \nabla \Phi(x(t)), x(t) - \bar{z} \rangle \geq \rho \|\nabla \Phi(x(t))\|.$$

Combining this inequality with (22) (that we recall below)

$$\dot{\mathcal{E}}_{\alpha,g}(t) = \frac{4}{\alpha-1} t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) - 2t \langle x(t) - \bar{z}, \nabla \Phi(x(t)) \rangle$$

we obtain

$$(54) \quad \dot{\mathcal{E}}_{\alpha,g}(t) + 2\rho t \|\nabla \Phi(x(t))\| \leq \frac{4}{\alpha-1} t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi).$$

Let us return to (23), which after integration, and using $\alpha > 3$, gives

$$\int_{t_0}^{\infty} t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) dt < +\infty.$$

As a consequence, by integrating (54), we deduce that

$$\int_{t_0}^{\infty} t \|\nabla \Phi(x(t))\| dt < +\infty.$$

By setting $f(t) = tg(t) - t\nabla \Phi(x(t))$, we can rewrite equation (1) as

$$t\ddot{x}(t) + \alpha\dot{x}(t) = f(t).$$

Since all assumptions of Lemma 3.3 are satisfied, we can affirm that $x(t)$ converges strongly to some $x^* \in \mathcal{H}$. Recalling that $\Phi(x(t)) \rightarrow \inf_{\mathcal{H}} \Phi$ and that Φ is continuous, we obtain $x^* \in \text{argmin } \Phi$. \square

Strong convergence in the case of an even function.

Recall that $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ is an even function if $\Phi(-x) = \Phi(x)$ for all $x \in \mathcal{H}$. In this case, $0 \in \text{argmin}_{\mathcal{H}} \Phi$.

Theorem 3.3. *Suppose that $\alpha > 3$, $\int_{t_0}^{+\infty} tg(t)dt < +\infty$, and Φ is an even function. Let $x(\cdot)$ be a classical global solution of equation (1). Then, there exists some $\bar{x} \in \text{argmin}_{\mathcal{H}} \Phi$ such that $x(t)$ converges strongly to \bar{x} as $t \rightarrow +\infty$.*

Proof. Set, for $t_0 \leq \tau \leq r$,

$$y(\tau) = \|x(\tau)\|^2 - \|x(r)\|^2 - \frac{1}{2}\|x(\tau) - x(r)\|^2.$$

By derivating twice, we obtain

$$\dot{y}(\tau) = \langle \dot{x}(\tau), x(\tau) + x(r) \rangle$$

and

$$\ddot{y}(\tau) = \|\dot{x}(\tau)\|^2 + \langle \ddot{x}(\tau), x(\tau) + x(r) \rangle.$$

From these two equations and (1), we deduce that

$$(55) \quad \begin{aligned} \ddot{y}(\tau) + \frac{\alpha}{\tau} \dot{y}(\tau) &= \|\dot{x}(\tau)\|^2 + \langle \ddot{x}(\tau) + \frac{\alpha}{\tau} \dot{x}(\tau), x(\tau) + x(r) \rangle \\ &= \|\dot{x}(\tau)\|^2 + \langle g(\tau) - \nabla \Phi(x(\tau)), x(\tau) + x(r) \rangle. \end{aligned}$$

Let us now consider the energy function, $W(\tau) = \frac{1}{2}\|\dot{x}(\tau)\|^2 + \Phi(x(\tau)) + \int_{\tau}^{\infty} \langle \dot{x}(t), g(t) \rangle dt$. We have $\frac{d}{d\tau} W(\tau) = -\frac{\alpha}{\tau} \|\dot{x}(\tau)\|^2$, and therefore W is a nonincreasing function. As a consequence, $W(\tau) \geq W(r)$, which equivalently gives

$$\frac{1}{2}\|\dot{x}(\tau)\|^2 + \Phi(x(\tau)) \geq \frac{1}{2}\|\dot{x}(r)\|^2 + \Phi(x(r)) - \int_{\tau}^r \langle \dot{x}(t), g(t) \rangle dt.$$

Using the convex differential inequality $\Phi(-x(r)) \geq \Phi(x(\tau)) - \langle \nabla \Phi(x(\tau)), x(\tau) + x(r) \rangle$, and the even property of Φ , $\Phi(x(r)) = \Phi(-x(r))$, we deduce that

$$\frac{1}{2}\|\dot{x}(\tau)\|^2 \geq -\langle \nabla \Phi(x(\tau)), x(\tau) + x(r) \rangle - \int_{\tau}^r \langle \dot{x}(t), g(t) \rangle dt.$$

Returning to (55), we finally obtain

$$\ddot{y}(\tau) + \frac{\alpha}{\tau} \dot{y}(\tau) \leq \frac{3}{2}\|\dot{x}(\tau)\|^2 + \langle g(\tau), x(\tau) + x(r) \rangle + \int_{\tau}^r \langle \dot{x}(t), g(t) \rangle dt.$$

Let us recall that, by Theorem 3.1, the trajectory $x(\cdot)$ is converging weakly, and hence bounded. Moreover, by (34), we have $\|\dot{x}(t)\| \leq \frac{C}{t}$. Hence, for some constant C

$$(56) \quad \ddot{y}(\tau) + \frac{\alpha}{\tau} \dot{y}(\tau) \leq k(\tau) := \frac{3}{2} \|\dot{x}(\tau)\|^2 + C \|g(\tau)\| + C \int_{\tau}^{+\infty} \frac{1}{t} \|g(t)\| dt.$$

Let us observe that the function k does not depend on r . Let us verify that $\tau \mapsto \tau k(\tau) \in L^1(t_0, +\infty)$. By Theorem 3.1, we have $\int_{t_0}^{\infty} t \|\dot{x}(t)\|^2 dt < +\infty$. By assumption, $\int_{t_0}^{+\infty} t g(t) dt < +\infty$. Moreover, by Fubini theorem

$$\int_{t_0}^{\infty} \tau \int_{\tau}^{\infty} \frac{1}{t} \|g(t)\| dt d\tau \leq \frac{1}{2} \int_{t_0}^{\infty} t \|g(t)\| dt < +\infty.$$

By integration of (56), by a similar argument as in Lemma 3.2, we obtain

$$(57) \quad \dot{y}(\tau) \leq \frac{C}{\tau^{\alpha}} + \frac{1}{\tau^{\alpha}} \int_{t_0}^{\tau} u^{\alpha} k(u) du,$$

where $C = t_0^{\alpha} \|\dot{x}(t_0)\| \|x\|_{\infty}$. Set

$$K(t) := \frac{C}{\tau^{\alpha}} + \frac{1}{\tau^{\alpha}} \int_{t_0}^{\tau} u^{\alpha} k(u) du.$$

By using Fubini theorem once more, and the fact that $\tau \mapsto \tau k(\tau) \in L^1(t_0, +\infty)$, we deduce that $K \in L^1(t_0, +\infty)$. Integrating $\dot{y}(\tau) \leq K(\tau)$ from t to r , we obtain

$$\frac{1}{2} \|x(t) - x(r)\|^2 \leq \|x(t)\|^2 - \|x(r)\|^2 + \int_t^r K(\tau) d\tau.$$

Since Φ is even, we have $0 \in \operatorname{argmin} \Phi$. Hence $\lim_{t \rightarrow +\infty} \|x(t)\|^2$ exists (see the proof of Theorem 3.1). As a consequence, $x(t)$ has the Cauchy property as $t \rightarrow +\infty$, and hence converges. \square

4. THE CASE $\operatorname{argmin} \Phi = \emptyset$.

Theorem 4.1. *Suppose $\alpha > 0$, $\int_{t_0}^{+\infty} \|g(t)\| dt < +\infty$, and $\inf \Phi > -\infty$. Then, for any orbit $x : [t_0, +\infty[\rightarrow \mathcal{H}$ of $(\text{AVD})_{\alpha, g}$, the following minimizing property holds*

$$\lim_{t \rightarrow +\infty} \Phi(x(t)) = \inf_{\mathcal{H}} \Phi.$$

We will use the following lemma, see [15].

Lemma 4.1. *Take $\delta > 0$, and let $f \in L^1(\delta, +\infty)$ be nonnegative. Consider a nondecreasing continuous function $\psi : (\delta, +\infty) \rightarrow (0, +\infty)$ such that $\lim_{t \rightarrow +\infty} \psi(t) = +\infty$. Then,*

$$\lim_{t \rightarrow +\infty} \frac{1}{\psi(t)} \int_{\delta}^t \psi(s) f(s) ds = 0.$$

Proof of Theorem 4.1. Let us first return to the proof of the energy estimates in Proposition 2.1. Replacing $\inf \Phi$ by $\min \Phi$ in the expression of the energy function, we obtain by the same argument

$$(58) \quad \sup_t \|\dot{x}(t)\| < +\infty,$$

$$(59) \quad \int_{t_0}^{+\infty} \frac{1}{t} \|\dot{x}(t)\|^2 dt < +\infty.$$

Consider the function $h(t) = \frac{1}{2} \|x(t) - z\|^2$, where this time, z is an arbitrary element of \mathcal{H} . We can easily verify that

$$\ddot{h}(t) + \frac{\alpha}{t} \dot{h}(t) = \|\dot{x}(t)\|^2 - \langle \nabla \Phi(x(t)), x(t) - z \rangle + \langle g(t), x(t) - z \rangle.$$

By convexity of Φ , we obtain

$$(60) \quad \ddot{h}(t) + \frac{\alpha}{t} \dot{h}(t) + \Phi(x(t)) - \Phi(z) \leq \|\dot{x}(t)\|^2 + \langle g(t), x(t) - z \rangle.$$

Consider the energy function

$$W(t) = \frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) - \inf \Phi + \int_t^{\infty} \langle \dot{x}(s), g(s) \rangle ds.$$

By classical derivation rules, and (1)

$$\begin{aligned} \frac{d}{dt} W(t) &= \langle \dot{x}(t), \ddot{x}(t) + \nabla \Phi(x(t)) - g(t) \rangle \\ &= -\frac{\alpha}{t} \|\dot{x}(t)\|^2 \leq 0. \end{aligned}$$

As a consequence, W is a nonincreasing function. Moreover, W is minorized by $-\|\dot{x}\|_{L^\infty} \int_{t_0}^{+\infty} \|g(s)\| ds$. Hence, there exists some $W_\infty \in \mathbb{R}$ such that $W(t) \rightarrow W_\infty$ as $t \rightarrow \infty$. Let us take advantage of this property, and reformulate (60) with the help of W :

$$(61) \quad \ddot{h}(t) + \frac{\alpha}{t} \dot{h}(t) + W(t) + \inf \Phi - \Phi(z) \leq \frac{3}{2} \|\dot{x}(t)\|^2 + \langle g(t), x(t) - z \rangle + \int_t^\infty \langle \dot{x}(s), g(s) \rangle ds.$$

For every $t > 0$, $W(t) \geq W_\infty$. Setting $B_\infty = W_\infty + \inf \Phi - \Phi(z)$, we obtain

$$B_\infty \leq \frac{3}{2} \|\dot{x}(t)\|^2 + \|g(t)\| \|x(t) - z\| + \|\dot{x}\|_{L^\infty} \int_t^\infty \|g(s)\| ds - \frac{1}{t^\alpha} \frac{d}{dt} (t^\alpha \dot{h}(t)).$$

Multiplying this last equation by $\frac{1}{t}$, and integrating between two reals $0 < t_0 < \theta$, we get

$$(62) \quad B_\infty \ln\left(\frac{\theta}{t_0}\right) \leq \frac{3}{2} \int_{t_0}^\theta \frac{1}{t} \|\dot{x}(t)\|^2 dt + \int_{t_0}^\theta \frac{\|g(t)\| \|x(t) - z\|}{t} dt + \|\dot{x}\|_{L^\infty} \int_{t_0}^\theta \left(\frac{1}{t} \int_t^\infty \|g(s)\| ds \right) dt - \int_{t_0}^\theta \frac{1}{t^{\alpha+1}} \frac{d}{dt} (t^\alpha \dot{h}(t)) dt.$$

Let us estimate the integrals in the second member of (62):

(1) By (58), $\int_{t_0}^{+\infty} \frac{1}{t} \|\dot{x}(t)\|^2 dt < +\infty$.

(2) Exploiting the relation $\|x(t) - z\| \leq \|x(t_0) - z\| + \int_{t_0}^t \|\dot{x}(s)\| ds$, we obtain

$$\int_{t_0}^\theta \frac{\|g(t)\| \|x(t) - z\|}{t} dt \leq \left(\frac{\|x_0 - z\|}{t_0} + \|\dot{x}\|_{L^\infty} \right) \int_{t_0}^{+\infty} \|g(t)\| dt < +\infty.$$

(3) After integration by parts

$$\int_{t_0}^\theta \left(\frac{1}{t} \int_t^\infty \|g(s)\| ds \right) dt = \ln \theta \int_\theta^\infty \|g(s)\| ds - \ln t_0 \int_{t_0}^\infty \|g(s)\| ds + \int_{t_0}^\theta \|g(t)\| \ln t dt.$$

(4) Set $I = \int_{t_0}^\theta \frac{1}{t^{\alpha+1}} \frac{d}{dt} (t^\alpha \dot{h}(t)) dt$. By integrating by parts twice

$$\begin{aligned} I &= \left[\frac{1}{t} \dot{h}(t) \right]_{t_0}^\theta + (\alpha + 1) \int_{t_0}^\theta \frac{1}{t^2} \dot{h}(t) dt \\ &= C + \frac{1}{\theta} \dot{h}(\theta) + \frac{(1+\alpha)}{\theta^2} h(\theta) + 2(1+\alpha) \int_{t_0}^\theta \frac{1}{t^3} h(t) dt. \end{aligned}$$

Since $h \geq 0$, we have $-I \leq -C - \frac{1}{\theta} \dot{h}(\theta)$. Then notice that $|\dot{h}(\theta)| = |\langle \dot{x}(\theta), x(\theta) - z \rangle| \leq \|\dot{x}\|_{L^\infty} (\|x(0) - z\| + \theta \|\dot{x}\|_{L^\infty})$.

Collecting the above results, we deduce from (62) that

$$(63) \quad B_\infty \ln\left(\frac{\theta}{t_0}\right) \leq C + \ln \theta \int_\theta^\infty \|g(s)\| ds + \|\dot{x}\|_{L^\infty} \int_{t_0}^\theta \|g(t)\| \ln t dt.$$

Dividing by $\ln(\frac{\theta}{t_0})$, and letting $\theta \rightarrow +\infty$, thanks to Lemma 4.1 with $\psi(t) = \ln t$, we conclude that $B_\infty \leq 0$. Equivalently, for every $z \in \mathcal{H}$, $W_\infty \leq \Phi(z) - \inf \Phi$, which leads to $W_\infty \leq 0$.

On the other hand, it is easy to see that $W(t) \geq \Phi(x(t)) - \inf \Phi - \|\dot{x}\|_{L^\infty} \int_t^{+\infty} g(s) ds$. Passing to the limit, as $t \rightarrow +\infty$, we deduce that

$$0 \geq W_\infty \geq \limsup \Phi(x(t)) - \inf \Phi.$$

Since we always have $\inf \Phi \leq \liminf \Phi(x(t))$, we conclude that $\lim_{t \rightarrow +\infty} \Phi(x(t)) = \inf \Phi$. \square

Remark 4.1. In [14], in the unperturbed case $g = 0$, it has been observed that, when $\operatorname{argmin} \Phi = \emptyset$, the fast convergence property of the values, as given in Theorem 2.1, may fail to be satisfied. A fortiori, without making additional assumption on the perturbation term, we also loose the fast convergence property in the perturbed case (take $g = 0!$).

5. FROM CONTINUOUS TO DISCRETE DYNAMICS AND ALGORITHMS

Time discretization of dissipative gradient-based dynamical systems leads naturally to algorithms, which, under appropriate assumptions, have similar convergence properties. This approach has been followed successfully in a variety of situations. For a general abstract discussion see [7], [8], and in the case of dynamics with inertial features see [3], [4], [6], [14], [15]. To cover practical situations involving constraints and/or nonsmooth data, we need to broaden our scope. This leads us to consider the non-smooth structured convex minimization problem

$$(64) \quad \min \{ \Phi(x) + \Psi(x) : x \in \mathcal{H} \}$$

where

- $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous proper function (which possibly takes the value $+\infty$);
- $\Psi : \mathcal{H} \rightarrow \mathbb{R}$ is a convex continuously differentiable function, whose gradient is Lipschitz continuous.

The optimal solutions of (64) satisfy

$$\partial \Phi(x) + \nabla \Psi(x) \ni 0,$$

where $\partial\Phi$ is the subdifferential of Φ in the sense of convex analysis. In order to adapt our dynamic to this non-smooth situation, we will consider the corresponding differential inclusion

$$(65) \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \partial\Phi(x(t)) + \nabla\Psi(x(t)) \ni g(t).$$

This dynamic is within the following framework

$$(66) \quad \ddot{x}(t) + a(t)\dot{x}(t) + \partial\Theta(x(t)) \ni g(t),$$

where $\Theta : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous proper function, and $a(\cdot)$ is a positive damping parameter.

The detailed study of this differential inclusion goes far beyond the scope of the present article, see [10] for some results in the case of a fixed positive damping parameter, i.e., $a(t) = \gamma > 0$ fixed, and $g = 0$. A formal analysis of this sytem shows that the Lyapunov analysis, which has been developed in the previous sections, still holds, as long as one does not use the Lipschitz continuity property of the gradient (cocoercivity property). This is based on the fact that the convexity (subdifferential) inequalities are still valid, as well as the (generalized) derivation chain rule, see [20]. Thus, setting $\Theta(x) = \Phi(x) + \Psi(x)$, we can reasonably assume that, for $\alpha > 3$, and $\int_{t_0}^{+\infty} t\|g(t)\|dt < +\infty$, for each trajectory of (65), there is rapid convergence of the values,

$$\Theta(x(t)) - \min \Theta \leq \frac{C}{t^2},$$

and weak convergence of the trajectory to an optimal solution.

Indeed, we are going to use these ideas as a guideline, and so introduce corresponding fast converging algorithms, making the link with Nesterov [31]-[34], Beck-Teboulle [19], and so extending the recent works of Chambolle-Dossal [26], Su-Boyd-Candès [41], Attouch-Peypouquet-Redont [14] to the perturbed case. As a basic ingredient of the discretization procedure, in order to preserve the fast convergence properties of the dynamical system (65), we are going to discretize it *implicitly* with respect to the nonsmooth function Φ , and *explicitly* with respect to the smooth function Ψ .

Taking a fixed time step size $h > 0$, and setting $t_k = kh$, $x_k = x(t_k)$ the implicit/explicit finite difference scheme for (65) gives

$$(67) \quad \frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{kh^2}(x_k - x_{k-1}) + \partial\Phi(x_{k+1}) + \nabla\Psi(y_k) \ni g_k,$$

where y_k is a linear combination of x_k and x_{k-1} , that will be made precise further. After developing (67), we obtain

$$(68) \quad x_{k+1} + h^2\partial\Phi(x_{k+1}) \ni x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) - h^2\nabla\Psi(y_k) + h^2g_k.$$

A natural choice for y_k leading to a simple formulation of the algorithm (other choices are possible, offering new directions of research for the future) is

$$(69) \quad y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}).$$

Using the classical proximal operator (equivalently, the resolvent operator of the maximal monotone operator $\partial\Phi$)

$$(70) \quad \text{prox}_{\gamma\Phi}(x) = \operatorname{argmin}_{\xi \in \mathcal{H}} \left\{ \Phi(\xi) + \frac{1}{2\gamma}\|\xi - x\|^2 \right\} = (I + \gamma\partial\Phi)^{-1}(x)$$

and setting $s = h^2$, the algorithm can be written as

$$(71) \quad \begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{s\Phi}(y_k - s(\nabla\Psi(y_k) - g_k)). \end{cases}$$

For practical purpose, and in order to fit with the existing litterature on the subject, it is convenient to work with the following equivalent formulation

$$(72) \quad (\text{AVD})_{\alpha, g} - \text{algo} \quad \begin{cases} y_k = x_k + \frac{k-1}{k+\alpha-1}(x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{s\Phi}(y_k - s(\nabla\Psi(y_k) - g_k)). \end{cases}$$

Indeed, we have $\frac{k-1}{k+\alpha-1} = 1 - \frac{\alpha}{k+\alpha-1}$. When α is an integer, up to the reindexation $k \mapsto k + \alpha - 1$, we obtain the same sequences (x_k) and (y_k) . For general $\alpha > 0$, we can easily verify that the algorithm $(\text{AVD})_{\alpha, g} - \text{algo}$ is still associated with the dynamical system (65).

This algorithm is within the scope of the proximal-based inertial algorithms [4], [30], [39], and forward-backward methods. In the unperturbed case, $g_k = 0$, it has been recently considered by Chambolle-Dossal [26], Su-Boyd-Candès [41], and Attouch-Peypouquet-Redont [14]. It enjoys fast convergence properties which are very similar to that of the continuous dynamic.

For $\alpha = 3$, $g_k = 0$, we recover the classical algorithm based on Nesterov and Güler ideas, and developed by Beck-Teboulle (FISTA)

$$(73) \quad \begin{cases} y_k = x_k + \frac{k-1}{k+2}(x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{s\Phi}(y_k - s\nabla\Psi(y_k)). \end{cases}$$

An important question regarding the (FISTA) method, as described in (73), is the convergence of sequences (x_k) and (y_k) . Indeed, it is still an open question. A major interest to consider the broader context of $(\text{AVD})_{\alpha,g}$ - algo algorithms is that, for $\alpha > 3$, these sequences converge, and they allow errors/perturbations, and using approximation methods. We will see that the proof of the convergence properties of $(\text{AVD})_{\alpha,g}$ - algo algorithms can be obtained in a parallel way with the convergence analysis in the continuous case in Theorem 3.1.

Theorem 5.1. *Let $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous proper function, and $\Psi : \mathcal{H} \rightarrow \mathbb{R}$ a convex continuously differentiable function, whose gradient is L -Lipschitz continuous. Suppose that $S = \text{argmin}(\Phi + \Psi)$ is nonempty. Suppose that $\alpha \geq 3$, $0 < s < \frac{1}{L}$, and $\sum_{k \in \mathbb{N}} k \|g_k\| < +\infty$. Let (x_k) be a sequence generated by the algorithm $(\text{AVD})_{\alpha,g}$ - algo. Then,*

$$(\Phi + \Psi)(x_k) - \min(\Phi + \Psi) = \mathcal{O}\left(\frac{1}{k^2}\right).$$

Precisely,

$$(74) \quad (\Phi + \Psi)(x_k) - \min(\Phi + \Psi) \leq \frac{C(\alpha - 1)}{2s(k + \alpha - 2)^2},$$

with C given by

$$C = \mathcal{G}(0) + 2s \left(\sum_{j=0}^{\infty} (j + \alpha - 1) \|g_j\| \right) \left(\sqrt{\frac{\mathcal{E}(0)}{\alpha - 1}} + \frac{2s}{\alpha - 1} \sum_{j=0}^{\infty} (j + \alpha - 1) \|g_j\| \right),$$

where

$$\mathcal{G}(0) = \frac{2s}{\alpha - 1} (\alpha - 2)^2 (\Theta(x_0) - \Theta^*) + (\alpha - 1) \|y_0 - x^*\|^2.$$

Proof. To simplify notations, we set $\Theta = \Phi + \Psi$, and take $x^* \in \text{argmin}\Theta$, i.e., $\Theta(x^*) = \inf \Theta$. In a parallel way to the continuous case, our proof is based on proving that $(\mathcal{E}(k))$ is a non-increasing sequence, where $\mathcal{E}(k)$ is the discrete version of the Lyapunov function $\mathcal{E}_{\alpha,g}(t)$ (we shall justify further that it is well defined), and which is given by

$$(75) \quad \mathcal{E}(k) := \frac{2s}{\alpha - 1} (k + \alpha - 2)^2 (\Theta(x_k) - \Theta(x^*) + (\alpha - 1) \|z_k - x^*\|^2) + \sum_{j=k}^{\infty} 2s (j + \alpha - 1) \langle g_j, z_{j+1} - x^* \rangle,$$

with

$$(76) \quad z_k := \frac{k + \alpha - 1}{\alpha - 1} y_k - \frac{k}{\alpha - 1} x_k.$$

In the passage from the continuous to discrete, we recall that we must use the reindexing $k \mapsto k + \alpha - 1$. Note that $\mathcal{E}(k)$ is equal to the Lyapunov function considered by Su-Boyd-Candès in [41, Theorem 4.3], plus a perturbation term. Let us introduce the function $\Psi_k : \mathcal{H} \rightarrow \mathbb{R}$ which is defined by

$$\forall y \in \mathcal{H}, \quad \Psi_k(y) := \Psi(y) - \langle g_k, y \rangle.$$

We also set

$$\Theta_k = \Phi + \Psi_k.$$

We have $\nabla\Psi_k(y) = \nabla\Psi(y) - g_k$, and hence $\nabla\Psi_k$ is still L -Lipschitz continuous. We can reformulate our algorithm with the help of Ψ_k as follows

$$(77) \quad (\text{AVD})_{\alpha,g} - \text{algo} \quad \begin{cases} y_k = x_k + \frac{k-1}{k+\alpha-1}(x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{s\Phi}(y_k - s\nabla\Psi_k(y_k)). \end{cases}$$

In order to analyze the convergence properties of the above algorithm, it is convenient to introduce the operator $G_{s,k} : \mathcal{H} \rightarrow \mathcal{H}$ which is defined by, for all $y \in \mathcal{H}$,

$$G_{s,k}(y) = \frac{1}{s} (y - \text{prox}_{s\Phi}(y - s\nabla\Psi_k(y))).$$

Equivalently,

$$\text{prox}_{s\Phi}(y - s\nabla\Psi_k(y)) = y - sG_{s,k}(y),$$

and the algorithm (77) can be formulated as

$$(78) \quad (\text{AVD})_{\alpha, g} - \text{algo} \quad \begin{cases} y_k = x_k + \frac{k-1}{k+\alpha-1}(x_k - x_{k-1}); \\ x_{k+1} = y_k - sG_{s,k}(y_k). \end{cases}$$

The variable z_k , which is defined in (76) by $z_k = \frac{k+\alpha-1}{\alpha-1}y_k - \frac{k}{\alpha-1}x_k$, will play an important role. It comes naturally into play as a discrete version of the term $\frac{t}{\alpha-1}\dot{x}(t) + x(t) - x^*$ which enters $\mathcal{E}_{\alpha, g}(t)$. Indeed,

$$(79) \quad \frac{k+\alpha-1}{\alpha-1}(x_{k+1} - x_k) + x_k = \frac{k+\alpha-1}{\alpha-1}x_{k+1} - \frac{k}{\alpha-1}x_k = z_{k+1}$$

where the last equality comes from (80) below. Let us examine the recursive relation satisfied by z_k . We have

$$(80) \quad \begin{aligned} z_{k+1} &= \frac{k+\alpha}{\alpha-1}y_{k+1} - \frac{k+1}{\alpha-1}x_{k+1} \\ &= \frac{k+\alpha}{\alpha-1} \left(x_{k+1} + \frac{k}{k+\alpha}(x_{k+1} - x_k) \right) - \frac{k+1}{\alpha-1}x_{k+1} \\ &= \frac{k+\alpha-1}{\alpha-1}x_{k+1} - \frac{k}{\alpha-1}x_k \\ &= \frac{k+\alpha-1}{\alpha-1}(y_k - sG_{s,k}(y_k)) - \frac{k}{\alpha-1}x_k \end{aligned}$$

$$(81) \quad = z_k - \frac{s}{\alpha-1}(k+\alpha-1)G_{s,k}(y_k).$$

We now use the classical formula in the proximal gradient (also called forward-backward) analysis (see [19], [26], [37], [41]): for any $x, y \in \mathcal{H}$

$$(82) \quad \Theta_k(y - sG_{s,k}(y)) \leq \Theta_k(x) + \langle G_{s,k}(y), y - x \rangle - \frac{s}{2}\|G_{s,k}(y)\|^2.$$

Note that this formula is valid since $s \leq \frac{1}{L}$, and $\nabla\Psi_k$ is L -lipschitz continuous. Let us write successively this formula at $y = y_k$ and $x = x_k$, then at $y = y_k$ and $x = x^*$. We obtain

$$\begin{aligned} \Theta_k(y_k - sG_{s,k}(y_k)) &\leq \Theta_k(x_k) + \langle G_{s,k}(y_k), y_k - x_k \rangle - \frac{s}{2}\|G_{s,k}(y_k)\|^2 \\ \Theta_k(y_k - sG_{s,k}(y_k)) &\leq \Theta_k(x^*) + \langle G_{s,k}(y_k), y_k - x^* \rangle - \frac{s}{2}\|G_{s,k}(y_k)\|^2. \end{aligned}$$

Multiplying the first equation by $\frac{k}{k+\alpha-1}$, and the second by $\frac{\alpha-1}{k+\alpha-1}$, then adding the two resulting equations, and using $x_{k+1} = y_k - sG_{s,k}(y_k)$, we obtain

$$(83) \quad \Theta_k(x_{k+1}) \leq \frac{k}{k+\alpha-1}\Theta_k(x_k) + \frac{\alpha-1}{k+\alpha-1}\Theta_k(x^*) - \frac{s}{2}\|G_{s,k}(y_k)\|^2$$

$$(84) \quad + \left\langle G_{s,k}(y_k), \frac{k}{k+\alpha-1}(y_k - x_k) + \frac{\alpha-1}{k+\alpha-1}(y_k - x^*) \right\rangle.$$

Let us rewrite the scalar product in (84) as follows:

$$(85) \quad \begin{aligned} \left\langle G_{s,k}(y_k), \frac{k}{k+\alpha-1}(y_k - x_k) + \frac{\alpha-1}{k+\alpha-1}(y_k - x^*) \right\rangle &= \frac{\alpha-1}{k+\alpha-1} \left\langle G_{s,k}(y_k), \frac{k}{\alpha-1}(y_k - x_k) + y_k - x^* \right\rangle \\ &= \frac{\alpha-1}{k+\alpha-1} \left\langle G_{s,k}(y_k), \frac{k+\alpha-1}{\alpha-1}y_k - \frac{k}{\alpha-1}x_k - x^* \right\rangle \\ &= \frac{\alpha-1}{k+\alpha-1} \langle G_{s,k}(y_k), z_k - x^* \rangle. \end{aligned}$$

Combining (83)-(84) with (85), we obtain

$$(86) \quad \Theta_k(x_{k+1}) \leq \frac{k}{k+\alpha-1}\Theta_k(x_k) + \frac{\alpha-1}{k+\alpha-1}\Theta_k(x^*) + \frac{\alpha-1}{k+\alpha-1} \langle G_{s,k}(y_k), z_k - x^* \rangle - \frac{s}{2}\|G_{s,k}(y_k)\|^2.$$

In order to write (86) in a recursive form, we use the relation (81) satisfied by z_k , which gives

$$z_{k+1} - x^* = z_k - x^* - \frac{s}{\alpha-1}(k+\alpha-1)G_{s,k}(y_k).$$

After developing

$$\|z_{k+1} - x^*\|^2 = \|z_k - x^*\|^2 - 2\frac{s}{\alpha-1}(k+\alpha-1) \langle z_k - x^*, G_{s,k}(y_k) \rangle + \frac{s^2}{(\alpha-1)^2}(k+\alpha-1)^2 \|G_{s,k}(y_k)\|^2,$$

and multiplying the above expression by $\frac{(\alpha-1)^2}{2s(k+\alpha-1)^2}$, we obtain

$$\frac{(\alpha-1)^2}{2s(k+\alpha-1)^2} (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2) = \frac{\alpha-1}{k+\alpha-1} \langle G_{s,k}(y_k), z_k - x^* \rangle - \frac{s}{2} \|G_{s,k}(y_k)\|^2.$$

Replacing this expression in (86), we obtain

$$(87) \quad \Theta_k(x_{k+1}) \leq \frac{k}{k+\alpha-1} \Theta_k(x_k) + \frac{\alpha-1}{k+\alpha-1} \Theta_k(x^*) + \frac{(\alpha-1)^2}{2s(k+\alpha-1)^2} (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2).$$

Equivalently

$$(88) \quad \Theta_k(x_{k+1}) - \Theta_k(x^*) \leq \frac{k}{k+\alpha-1} (\Theta_k(x_k) - \Theta_k(x^*)) + \frac{(\alpha-1)^2}{2s(k+\alpha-1)^2} (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2).$$

Returning to $\Theta(y) = \Theta_k(y) + \langle g_k, y \rangle$, we obtain

$$(89) \quad \begin{aligned} \Theta(x_{k+1}) - \Theta(x^*) &\leq \frac{k}{k+\alpha-1} (\Theta(x_k) - \Theta(x^*)) + \frac{(\alpha-1)^2}{2s(k+\alpha-1)^2} (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2) \\ &\quad + \langle g_k, x_{k+1} - x^* \rangle - \frac{k}{k+\alpha-1} \langle g_k, x_k - x^* \rangle. \end{aligned}$$

After reduction

$$(90) \quad \begin{aligned} \Theta(x_{k+1}) - \Theta(x^*) &\leq \frac{k}{k+\alpha-1} (\Theta(x_k) - \Theta(x^*)) + \frac{(\alpha-1)^2}{2s(k+\alpha-1)^2} (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2) \\ &\quad + \left\langle g_k, x_{k+1} - x_k + \frac{\alpha-1}{k+\alpha-1} (x_k - x^*) \right\rangle. \end{aligned}$$

Multiplying by $\frac{2s}{\alpha-1} (k+\alpha-1)^2$, we obtain

$$(91) \quad \begin{aligned} \frac{2s}{\alpha-1} (k+\alpha-1)^2 (\Theta(x_{k+1}) - \Theta(x^*)) &\leq \frac{2s}{\alpha-1} k (k+\alpha-1) (\Theta(x_k) - \Theta(x^*)) + (\alpha-1) (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2) \\ &\quad + \frac{2s}{\alpha-1} (k+\alpha-1)^2 \left\langle g_k, x_{k+1} - x_k + \frac{\alpha-1}{k+\alpha-1} (x_k - x^*) \right\rangle. \end{aligned}$$

For $\alpha \geq 3$ one can easily verify that

$$k(k+\alpha-1) \leq (k+\alpha-2)^2.$$

More precisely

$$k(k+\alpha-1) = (k+\alpha-2)^2 - k(\alpha-3) - (\alpha-2)^2 \leq (k+\alpha-2)^2 - k(\alpha-3).$$

As a consequence, from (91) we deduce that

$$(92) \quad \begin{aligned} \frac{2s}{\alpha-1} (k+\alpha-1)^2 (\Theta(x_{k+1}) - \Theta(x^*)) + 2s \frac{\alpha-3}{\alpha-1} k (\Theta(x_k) - \Theta(x^*)) &\leq \frac{2s}{\alpha-1} (k+\alpha-2)^2 (\Theta(x_k) - \Theta(x^*)) \\ &\quad + (\alpha-1) (\|z_k - x^*\|^2 - \|z_{k+1} - x^*\|^2) + \frac{2s}{\alpha-1} (k+\alpha-1)^2 \left\langle g_k, x_{k+1} - x_k + \frac{\alpha-1}{k+\alpha-1} (x_k - x^*) \right\rangle. \end{aligned}$$

Setting

$$(93) \quad \mathcal{G}(k) = \frac{2s}{\alpha-1} (k+\alpha-2)^2 (\Theta(x_k) - \Theta^*) + (\alpha-1) \|z_k - x^*\|^2,$$

we can reformulate (92) as

$$(94) \quad \mathcal{G}(k+1) + 2s \frac{\alpha-3}{\alpha-1} k (\Theta(x_k) - \Theta(x^*)) \leq \mathcal{G}(k) + \frac{2s}{\alpha-1} (k+\alpha-1)^2 \left\langle g_k, x_{k+1} - x_k + \frac{\alpha-1}{k+\alpha-1} (x_k - x^*) \right\rangle.$$

Equivalently

$$(95) \quad \mathcal{G}(k+1) + 2s \frac{\alpha-3}{\alpha-1} k (\Theta(x_k) - \Theta(x^*)) \leq \mathcal{G}(k) + 2s (k+\alpha-1) \left\langle g_k, \frac{k+\alpha-1}{\alpha-1} (x_{k+1} - x_k) + x_k - x^* \right\rangle.$$

Using (79)

$$z_{k+1} = \frac{k+\alpha-1}{\alpha-1} (x_{k+1} - x_k) + x_k,$$

we deduce that

$$(96) \quad \mathcal{G}(k+1) + 2s \frac{\alpha-3}{\alpha-1} k (\Theta(x_k) - \Theta(x^*)) \leq \mathcal{G}(k) + 2s (k+\alpha-1) \langle g_k, z_{k+1} - x^* \rangle.$$

We now develop a similar analysis as in the continuous case. Given some integer K , set

$$\mathcal{E}_K(k) = \mathcal{G}(k) + \sum_{j=k}^K 2s(j + \alpha - 1) \langle g_j, z_{j+1} - x^* \rangle.$$

Then (96) is equivalent to

$$(97) \quad \mathcal{E}_K(k+1) + 2s \frac{\alpha-3}{\alpha-1} k (\Theta(x_k) - \Theta(x^*)) \leq \mathcal{E}_K(k).$$

Hence, the sequence $(\mathcal{E}_K(k))$ is nonincreasing. In particular $\mathcal{E}_K(k) \leq \mathcal{E}_K(0)$, which gives

$$\mathcal{G}(k) + \sum_{j=k}^K 2s(j + \alpha - 1) \langle g_j, z_{j+1} - x^* \rangle \leq \mathcal{G}(0) + \sum_{j=0}^K 2s(j + \alpha - 1) \langle g_j, z_{j+1} - x^* \rangle.$$

As a consequence

$$(98) \quad \mathcal{G}(k) \leq \mathcal{G}(0) + \sum_{j=0}^{k-1} 2s(j + \alpha - 1) \langle g_j, z_{j+1} - x^* \rangle.$$

By definition of $\mathcal{G}(k)$, neglecting some positive terms, and by Cauchy-Schwarz inequality, we infer

$$(\alpha - 1) \|z_k - x^*\|^2 \leq \mathcal{G}(0) + 2s \sum_{j=0}^{k-1} (j + \alpha - 1) \|g_j\| \|z_{j+1} - x^*\|.$$

Equivalently

$$(99) \quad \|z_k - x^*\|^2 \leq \frac{1}{\alpha - 1} \mathcal{G}(0) + \frac{2s}{\alpha - 1} \sum_{j=1}^k (j + \alpha - 2) \|g_{j-1}\| \|z_j - x^*\|.$$

We then use the following result, a discrete version of Gronwall's lemma.

Lemma 5.1. *Let (a_k) be a sequence of positive real numbers such that*

$$a_k^2 \leq c + \sum_{j=1}^k \beta_j a_j$$

where (β_j) is a sequence of positive real numbers such that $\sum_j \beta_j < +\infty$, and c is a positive real number. Then

$$a_k \leq \sqrt{c} + \sum_{j=1}^{\infty} \beta_j.$$

Proof. Set $A_k := \sup_{1 \leq j \leq k} a_j$. Then, for $1 \leq l \leq k$

$$a_l^2 \leq c + \sum_{j=1}^l \beta_j a_j \leq c + A_k \sum_{j=1}^{\infty} \beta_j$$

Passing to the supremum with respect to l , with $1 \leq l \leq k$, we obtain

$$A_k^2 \leq c + A_k \sum_{j=1}^{\infty} \beta_j.$$

By elementary algebraic computation, it follows that

$$A_k \leq \sqrt{c} + \sum_{j=1}^{\infty} \beta_j.$$

□

Following the proof of Theorem 5.1. From (99), applying Lemma 5.1 with $a_k = \|z_k - x^*\|$, we deduce that

$$(100) \quad \|z_k - x^*\| \leq M := \sqrt{\frac{\mathcal{G}(0)}{\alpha - 1}} + \frac{2s}{\alpha - 1} \sum_{j=0}^{\infty} (j + \alpha - 1) \|g_j\|.$$

Note that M is finite, because of the assumption $\sum_{k \in \mathbb{N}} k \|g_k\| < +\infty$. Returning to (98) we obtain

$$(101) \quad \mathcal{G}(k) \leq C := \mathcal{G}(0) + 2s \left(\sum_{j=0}^{\infty} (j + \alpha - 1) \|g_j\| \right) \left(\sqrt{\frac{\mathcal{G}(0)}{\alpha - 1}} + \frac{2s}{\alpha - 1} \sum_{j=0}^{\infty} (j + \alpha - 1) \|g_j\| \right).$$

By definition of $\mathcal{G}(k)$, and the positivity of its constitutive elements we finally obtain

$$\frac{2s}{\alpha - 1} (k + \alpha - 2)^2 (\Theta(x_k) - \Theta^*) \leq C.$$

which gives (74). \square

Remark 5.1. In the particular case $\alpha = 3$, for a perturbed version of the classical FISTA algorithm, Schmidt, Le Roux, and Bach proved in [40] a result similar to Theorem 5.1 concerning the fast convergence of the values.

Let us now study the convergence of the sequence (x_k) .

Theorem 5.2. *Let $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous proper function, and $\Psi : \mathcal{H} \rightarrow \mathbb{R}$ a convex continuously differentiable function, whose gradient is L -Lipschitz continuous. Suppose that $S = \operatorname{argmin}(\Phi + \Psi)$ is nonempty. Suppose that $\alpha > 3$, $0 < s < \frac{1}{L}$, and $\sum_{k \in \mathbb{N}} k \|g_k\| < +\infty$. Let (x_k) be a sequence generated by the algorithm $(\text{AVD})_{\alpha, g}$ -algo. Then,*

- i) $\sum_k k ((\Phi + \Psi)(x_k) - \inf(\Phi + \Psi)) < +\infty$;
- ii) $\sum k \|x_{k+1} - x_k\|^2 < +\infty$;
- iii) (x_k) converges weakly, as $k \rightarrow +\infty$, to some $x^* \in \operatorname{argmin} \Phi$.

Proof. The demonstration is parallel to that of Theorem 3.1.

Step 1. Let us return to (96),

$$\mathcal{G}(k+1) + 2s \frac{\alpha - 3}{\alpha - 1} k (\Theta(x_k) - \Theta(x^*)) \leq \mathcal{G}(k) + 2s (k + \alpha - 1) \langle g_k, z_{k+1} - x^* \rangle.$$

By (100), we know that the sequence (z_k) is bounded. Summing the above inequalities, and using $\alpha > 3$, we obtain

$$(102) \quad \sum_k k ((\Phi + \Psi)(x_k) - \inf(\Phi + \Psi)) < +\infty,$$

thats' item i).

Step 2. Now apply the fundamental inequality (82), which can be equivalently written as follows

$$(103) \quad \Theta_k(y - sG_{s,k}(y)) + \frac{1}{2s} \|y - sG_{s,k}(y) - x\|^2 \leq \Theta_k(x) + \frac{1}{2s} \|x - y\|^2.$$

Take $y = y_k$, and $x = x_k$. Since $x_{k+1} = y_k - sG_{s,k}(y_k)$, and $y_k - x_k = \frac{k-1}{k+\alpha-1}(x_k - x_{k-1})$, we obtain

$$(104) \quad \Theta_k(x_{k+1}) + \frac{1}{2s} \|x_{k+1} - x_k\|^2 \leq \Theta_k(x_k) + \frac{1}{2s} \frac{(k-1)^2}{(k+\alpha-1)^2} \|x_k - x_{k-1}\|^2.$$

Equivalently, by definition of Θ_k ,

$$(105) \quad \Theta(x_{k+1}) + \frac{1}{2s} \|x_{k+1} - x_k\|^2 \leq \Theta(x_k) + \frac{1}{2s} \frac{(k-1)^2}{(k+\alpha-1)^2} \|x_k - x_{k-1}\|^2 + \langle g_k, x_{k+1} - x_k \rangle.$$

To shorten notations, set $\theta_k = \Theta(x_k) - \Theta(x^*)$, $d_k = \frac{1}{2} \|x_k - x_{k-1}\|^2$, $a = \alpha - 1$. By Cauchy-Schwarz inequality, and with these notations, (105) gives

$$(106) \quad \frac{1}{s} \left(d_{k+1} - \frac{(k-1)^2}{(k+a)^2} d_k \right) \leq (\theta_k - \theta_{k+1}) + \|g_k\| \|x_{k+1} - x_k\|.$$

After multiplication by $(k+a)^2$, we obtain

$$(107) \quad \frac{1}{s} ((k+a)^2 d_{k+1} - (k-1)^2 d_k) \leq (k+a)^2 (\theta_k - \theta_{k+1}) + (k+a)^2 \|g_k\| \|x_{k+1} - x_k\|.$$

Summing from $k = 1$ to $k = K$ gives

$$(108) \quad \sum_{k=1}^K ((k+a)^2 d_{k+1} - (k-1)^2 d_k) \leq s \sum_{k=1}^K (k+a)^2 (\theta_k - \theta_{k+1}) + s \sum_{k=1}^K (k+a)^2 \|g_k\| \|x_{k+1} - x_k\|.$$

By a similar computation as in Chambolle-Dossal [26, Corollary 2], we equivalently obtain

$$(109) \quad (K+a)^2 d_{K+1} + \sum_{k=2}^K a(2k+a-2) d_k \leq s \left((a+1)^2 \theta_1 - (K+a)^2 \theta_{K+1} + \sum_{k=2}^K (2k+2a-1) \theta_k + \sum_{k=1}^K (k+a)^2 \|g_k\| \|x_{k+1} - x_k\| \right).$$

By (102) we have $\sum_k (2k + 2a - 1) \theta_k < +\infty$. Hence there exists some constant C such that, for all $K \in \mathbb{N}$

$$(110) \quad (K + a)^2 \|x_{K+1} - x_K\|^2 \leq C + 2s \sum_{k=1}^K (k + a)^2 \|g_k\| \|x_{k+1} - x_k\|.$$

We now proceed to a parallel argument to that used in the proof of Theorem 3.1. Let us write (110) as follows, with $r_k := (k + a) \|x_{k+1} - x_k\|$

$$(111) \quad r_k^2 \leq C + 2s \sum_{j=1}^k (j + a) \|g_j\| r_j.$$

We make appeal to the following discrete version of the Gronwall-Bellman lemma.

Lemma 5.2. *Let (r_k) be sequence of positive real numbers such that, for all $k \geq 1$*

$$r_k^2 \leq C + \sum_{j=1}^k \omega_j r_j$$

where C is a positive constant, and $\sum_k \omega_j < +\infty$, with $\omega_j \geq 0$. Then the sequence (r_k) is bounded with

$$r_k \leq \sqrt{C} + \sum_{j \in \mathbb{N}} \omega_j.$$

Proof. For simplicity, let us assume $\omega_j > 0$ (one can always reduce to this situation by adding some positive constant, arbitrarily small, see Brezis [20] for the proof of this lemma in the continuous case). Set $A_k := C + \sum_{j=1}^k \omega_j r_j$, $A_0 = C$. We have $r_k^2 \leq A_k$, and $A_{k+1} - A_k = \omega_{k+1} r_{k+1}$. Equivalently $r_{k+1} = \frac{A_{k+1} - A_k}{\omega_{k+1}}$, which gives

$$\frac{A_{k+1} - A_k}{\omega_{k+1}} \leq \sqrt{A_{k+1}},$$

and hence

$$\frac{A_{k+1}}{\sqrt{A_{k+1}}} - \frac{A_k}{\sqrt{A_{k+1}}} \leq \omega_{k+1}.$$

From this, and using that the sequence (A_k) is increasing, we deduce that

$$\sqrt{A_{k+1}} - \sqrt{A_k} \leq \omega_{k+1}.$$

Summing this inequality, and using $r_k \leq \sqrt{A_k}$ gives the claim. \square

Following the proof of Theorem 5.2. Let us apply lemma 5.2 to inequality (111) with $r_j = (j + a) \|x_{j+1} - x_j\|$, and $\omega_j = (j + a) \|g_j\|$. By using the assumption on the perturbation term $\sum_k k \|g_k\| < +\infty$, we deduce that

$$(112) \quad \sup_k k \|x_{k+1} - x_k\| < +\infty.$$

Injecting this information in (109), we obtain

$$(113) \quad \sum_k a(2k + a - 2) d_k \leq C + \sum_k (2k + 2a - 1) \theta_k + \sup_k ((k + a) \|x_{k+1} - x_k\|) \sum_k (k + a) \|g_k\|.$$

From $a = \alpha - 1 \geq 2$, (102), and the definition of d_k , we deduce that

$$\sum_k k \|x_{k+1} - x_k\|^2 < +\infty,$$

which is our claim *ii*).

Step 3. The last step consists in applying Opial's lemma, whose discrete version is stated below.

Lemma 5.3. *Let S be a non empty subset of \mathcal{H} , and (x_k) a sequence of elements of \mathcal{H} . Assume that*

- (i) *for every $z \in S$, $\lim_{k \rightarrow +\infty} \|x_k - z\|$ exists;*
- (ii) *every weak sequential cluster point of the sequence (x_k) belongs to S .*

Then

$$w - \lim_{k \rightarrow +\infty} x_k = x_\infty \quad \text{exists, for some element } x_\infty \in S.$$

We are going to apply Opial's lemma with $S = \operatorname{argmin}(\Phi + \Psi)$. By Theorem 5.1, we have $(\Phi + \Psi)(x_k) \rightarrow \min(\Phi + \Psi)$ (indeed, we have proved fast convergence). By the lower semicontinuity property of $\Phi + \Psi$ for the weak convergence of \mathcal{H} , we immediately obtain that item (ii) of Opial's lemma is satisfied. Thus the only point to verify is that $\lim \|x_k - x^*\|$ exists for any $x^* \in \operatorname{argmin}(\Phi + \Psi)$. Equivalently, we are going to show that $\lim h_k$ exists, with $h_k := \frac{1}{2}\|x_k - x^*\|^2$.

The beginning of the proof is similar to [4], [26]. It consists in establishing a discrete version of the second-order differential inequality (52)

$$\ddot{h}(t) + \frac{\alpha}{t}\dot{h}(t) \leq \|\dot{x}(t)\|^2 + \|x(t) - x^*\|\|g(t)\|.$$

We use the parallelogram identity, which in an equivalent form can be written as follows: for any $a, b, c \in \mathcal{H}$

$$(114) \quad \frac{1}{2}\|a - b\|^2 + \frac{1}{2}\|a - c\|^2 = \frac{1}{2}\|b - c\|^2 + \langle a - b, a - c \rangle.$$

Taking $b = x^*$, $a = x_{k+1}$, $c = x_k$, we obtain

$$\frac{1}{2}\|x_{k+1} - x^*\|^2 + \frac{1}{2}\|x_{k+1} - x_k\|^2 = \frac{1}{2}\|x_k - x^*\|^2 + \langle x_{k+1} - x^*, x_{k+1} - x_k \rangle.$$

Equivalently,

$$(115) \quad h_k - h_{k+1} = \frac{1}{2}\|x_{k+1} - x_k\|^2 + \langle x_{k+1} - x^*, x_k - x_{k+1} \rangle.$$

By definition of y_k we have

$$x_k - x_{k+1} = y_k - x_{k+1} - \frac{k-1}{k+\alpha-1}(x_k - x_{k-1}).$$

Replacing in (115), we obtain

$$(116) \quad h_k - h_{k+1} = \frac{1}{2}\|x_{k+1} - x_k\|^2 + \langle x_{k+1} - x^*, y_k - x_{k+1} \rangle - \frac{k-1}{k+\alpha-1} \langle x_{k+1} - x^*, x_k - x_{k-1} \rangle.$$

Let us now use the monotonicity property of $\partial\Phi$. Since $-s\nabla\Psi(x^*) \in s\partial\Phi(x^*)$, and $y_k - x_{k+1} - s\nabla\Psi(y_k) + sg_k \in s\partial\Phi(x_{k+1})$, we have

$$\langle y_k - x_{k+1} - s\nabla\Psi(y_k) + sg_k + s\nabla\Psi(x^*), x_{k+1} - x^* \rangle \geq 0.$$

Equivalently

$$\langle y_k - x_{k+1}, x_{k+1} - x^* \rangle + s \langle \nabla\Psi(x^*) - \nabla\Psi(y_k) + g_k, x_{k+1} - x^* \rangle \geq 0.$$

Replacing in (116) we obtain

$$(117) \quad h_{k+1} - h_k + \frac{1}{2}\|x_{k+1} - x_k\|^2 + s \langle \nabla\Psi(y_k) - \nabla\Psi(x^*) - g_k, x_{k+1} - x^* \rangle - \frac{k-1}{k+\alpha-1} \langle x_{k+1} - x^*, x_k - x_{k-1} \rangle \leq 0.$$

We now use the co-coercivity of $\nabla\Psi$

$$(118) \quad \begin{aligned} \langle \nabla\Psi(y_k) - \nabla\Psi(x^*), x_{k+1} - x^* \rangle &= \langle \nabla\Psi(y_k) - \nabla\Psi(x^*), x_{k+1} - y_k \rangle + \langle \nabla\Psi(y_k) - \nabla\Psi(x^*), y_k - x^* \rangle \\ &\geq \frac{1}{L}\|\Psi(y_k) - \nabla\Psi(x^*)\|^2 + \langle \nabla\Psi(y_k) - \nabla\Psi(x^*), x_{k+1} - y_k \rangle \\ &\geq \frac{1}{L}\|\Psi(y_k) - \nabla\Psi(x^*)\|^2 - \|\nabla\Psi(y_k) - \nabla\Psi(x^*)\|\|x_{k+1} - y_k\| \\ &\geq -\frac{L}{2}\|x_{k+1} - y_k\|^2. \end{aligned}$$

Combining (117) and (118)

$$(119) \quad h_{k+1} - h_k + \frac{1}{2}\|x_{k+1} - x_k\|^2 - \frac{sL}{2}\|x_{k+1} - y_k\|^2 - s\|g_k\|\|x_{k+1} - x^*\| - \frac{k-1}{k+\alpha-1} \langle x_{k+1} - x^*, x_k - x_{k-1} \rangle \leq 0.$$

Let us use again (114) with $b = x^*$, $a = x_k$, $c = x_{k-1}$. We obtain

$$\frac{1}{2}\|x_k - x^*\|^2 + \frac{1}{2}\|x_k - x_{k-1}\|^2 = \frac{1}{2}\|x_{k-1} - x^*\|^2 + \langle x_k - x^*, x_k - x_{k-1} \rangle.$$

Equivalently

$$(120) \quad h_{k-1} - h_k = \frac{1}{2}\|x_k - x_{k-1}\|^2 - \langle x_k - x^*, x_k - x_{k-1} \rangle.$$

Combining (119) with (120) we obtain

$$(121) \quad \begin{aligned} h_{k+1} - h_k - \frac{k-1}{k+\alpha-1} (h_k - h_{k-1}) &\leq -\frac{1}{2}\|x_{k+1} - x_k\|^2 + \frac{sL}{2}\|x_{k+1} - y_k\|^2 + s\|g_k\|\|x_{k+1} - x^*\| \\ &\quad + \frac{k-1}{k+\alpha-1} \left(\frac{1}{2}\|x_k - x_{k-1}\|^2 + \langle x_k - x_{k-1}, x_{k+1} - x_k \rangle \right). \end{aligned}$$

By definition of $y_k = x_k + \frac{k-1}{k+\alpha-1}(x_k - x_{k-1})$, we have $x_{k+1} - y_k = x_{k+1} - x_k - \frac{k-1}{k+\alpha-1}(x_k - x_{k-1})$. Hence

$$\|x_{k+1} - y_k\|^2 = \|x_{k+1} - x_k\|^2 + \left(\frac{k-1}{k+\alpha-1}\right)^2 \|x_k - x_{k-1}\|^2 - 2\frac{k-1}{k+\alpha-1} \langle x_{k+1} - x_k, x_k - x_{k-1} \rangle$$

Substituting in (121), we obtain

$$(122) \quad h_{k+1} - h_k - \gamma_k(h_k - h_{k-1}) \leq -(1 - \frac{sL}{2})\|x_{k+1} - y_k\|^2 + s\|g_k\|\|x_{k+1} - x^*\| + (\gamma_k + \gamma_k^2)\|x_k - x_{k-1}\|^2,$$

where $\gamma_k = \frac{k-1}{k+\alpha-1}$. Since $0 < s < \frac{1}{L}$, we have $(1 - \frac{sL}{2}) > 0$. On the other hand, since $\gamma_k < 1$, we have $\gamma_k + \gamma_k^2 < 2\gamma_k$. Hence

$$(123) \quad h_{k+1} - h_k - \gamma_k(h_k - h_{k-1}) \leq s\|g_k\|\|x_{k+1} - x^*\| + 2\gamma_k\|x_k - x_{k-1}\|^2.$$

By (100), we know that the sequence (z_k) is bounded. By (112), we know that $\sup_k k\|x_{k+1} - x_k\| < +\infty$. Since $x_k = z_k - \frac{k+\alpha-1}{\alpha-1}(x_{k+1} - x_k)$, we deduce that the sequence (x_k) is bounded. Returning to (123), we have, for some constant C

$$(124) \quad h_{k+1} - h_k - \gamma_k(h_k - h_{k-1}) \leq C\|g_k\| + 2\gamma_k\|x_k - x_{k-1}\|^2.$$

We now use the estimation that we obtained in step 2, namely $\sum_k k\|x_{k+1} - x_k\|^2 < +\infty$. Combined with the assumption $\sum_k k\|g_k\| < +\infty$, we deduce that

$$(125) \quad h_{k+1} - h_k - \gamma_k(h_k - h_{k-1}) \leq \omega_k,$$

for some nonnegative sequence (ω_k) such that $\sum_{k \in \mathbb{N}} k\omega_k < +\infty$. Taking the positive part, we obtain

$$(126) \quad (h_{k+1} - h_k)^+ - \gamma_k(h_k - h_{k-1})^+ \leq \omega_k.$$

We are now using the following lemma, which is a discrete version of lemma 3.2.

Lemma 5.4. *Let (a_k) be sequence of nonnegative real numbers such that, for all $k \geq 1$*

$$a_{k+1} \leq \frac{k-1}{k+\alpha-1}a_k + \omega_k$$

where $\alpha \geq 3$, and $\sum_k k\omega_k < +\infty$, with $\omega_k \geq 0$. Then the sequence (a_k) is summable, i.e.,

$$\sum_{k \in \mathbb{N}} a_k < +\infty.$$

Proof. Since $\alpha \geq 3$ we have $\alpha - 1 \geq 2$, and hence

$$a_{k+1} \leq \frac{k-1}{k+2}a_k + \omega_k.$$

Multiplying this expression by $(k+1)^2$, we obtain

$$(k+1)^2 a_{k+1} \leq \frac{(k-1)(k+1)^2}{k+2}a_k + (k+1)^2 \omega_k.$$

Then note that, for all integer k

$$\frac{(k-1)(k+1)^2}{k+2} \leq k^2.$$

Hence

$$(k+1)^2 a_{k+1} \leq k^2 a_k + (k+1)^2 \omega_k.$$

Summing this inequality with respect to $j = 1, 2, \dots, k$, we obtain

$$k^2 a_k \leq a_1 + \sum_{j=1}^{k-1} (j+1)^2 \omega_j.$$

Dividing by k^2 , and summing with respect to k , we obtain

$$\sum_k a_k \leq a_1 \sum_k \frac{1}{k^2} + \sum_k \frac{1}{k^2} \sum_{j=1}^{k-1} (j+1)^2 \omega_j.$$

Applying Fubini theorem to this last sum, we obtain

$$\sum_k a_k \leq a_1 \sum_k \frac{1}{k^2} + \sum_j \left(\sum_{k=j+1}^{\infty} \frac{1}{k^2} \right) (j+1)^2 \omega_j.$$

We have

$$\sum_{k=j+1}^{\infty} \frac{1}{k^2} \leq \int_j^{\infty} \frac{1}{t^2} dt = \frac{1}{j}.$$

Hence

$$\sum_k a_k \leq a_1 \sum \frac{1}{k^2} + \sum_j \frac{(j+1)^2}{j} \omega_j < +\infty,$$

which by $\frac{(j+1)^2}{j} \leq 4j$ for $j \geq 1$ gives the claim. \square

End of the proof of Theorem 5.2. Let us apply lemma 5.4 with $a_k = (h_k - h_{k-1})^+$. We obtain

$$\sum_k (h_k - h_{k-1})^+ < +\infty,$$

which, combined with h_k nonnegative, gives the convergence of the sequence (h_k) , and ends the proof. \square

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